Math 104 004 **Final Exam** **December 15, 2021**

Name: ______________________________

SID: ______________________________

Instructions :

1. You have 170 minutes, 8:10am-11:00am. You may not need that much time.
2. No books, notes, or other outside materials are allowed.
3. There are 7 questions on the exam. Each question is worth 10 points.
4. You need to show all of your work and justify all statements. If you need more space, use the pages at the back of the exam or come get more paper at the front of the class. If you do so, please indicate which page your solution continues on.
5. Before you begin, take a quick look at all the questions on the exam, and start with the one you feel the most comfortable solving. It is more important to do the problems well that you know how to do, than it is to finish the whole exam.
6. While attempting any problem, do write something even if you are unable to solve it completely. You may get partial credit.

(Do not fill these in; they are for grading purposes only.)

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Total	

1. a) (5 pts) Let $f : [0,1] \to \mathbb{R}$ be an integrable function . Prove that there exists $x \in [0,1]$ such that

$$\int_0^x f = \frac{1}{3}\int_0^1 f.$$

b) (5 pts) Let $f : [0,1] \to \mathbb{R}$ be a continuous function . Prove that there exists $x \in [0,1]$ such that

$$f(x) = \int_0^1 f.$$

1) By the fundamentat theorem, $F : [0,1] \to \mathbb{R}$, $F(x) = \int_0^x f$ is continuous.

$F(0) = 0$ and $F(1) = \int_0^1 f$, so $\frac{1}{3}\int_0^1 f$ is between $F(0)$ and $F(1)$.

By the IVT, there exists $x \in [0,1]$ such that $\int_0^x f = F(x) = \frac{1}{3}\int_0^1 f$.

2) By the fundaental thm, F (as in part 1) is diff. on $(0,1)$ and $F' = f$. Since F is continuous, by the MVT there exists $x \in (0,1)$ such that

$$F'(x) = f(x) = \frac{F(1) - F(0)}{1 - 0} = \int_0^1 f.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(0) = 0$. Assume further that f' is differentiable at 0 and $f''(0) < 0$.

a) (5 pts) Prove that there exists $\delta > 0$ such that

$$\frac{f'(x)}{x} < 0$$

for all $x \in (-\delta, \delta)$, $x \neq 0$.

b) (5 pts) Prove that $f(0) > f(x)$ for all $x \in (-\delta, \delta)$, $x \neq 0$.

a) By definition, $\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = f''(0)$

Choose $\varepsilon = |f''(0)|$, which is > 0. There exists $\delta > 0$ such that $|x - 0| < \delta$, $x \neq 0$ implies

$$\left| \frac{f'(x) - f'(0)}{x - 0} - f''(0) \right| = \left| \frac{f'(x)}{x} - f''(0) \right| < \varepsilon = |f''(0)|.$$

Thus for $x \in (-\delta, \delta)$, $x \neq 0$,

$$\frac{f'(x)}{x} < f''(0) + |f''(0)| = 0.$$

b) Let $x \in (0, \delta)$. By the MVT there exists $y \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(y)$$

Since $y > 0$, and $\frac{f'(y)}{y} < 0$ by part a, $f'(y) < 0$.

So $\frac{f(x) - f(0)}{x} < 0$, and since $x > 0$, $f(x) - f(0) < 0$.

If $x \in (-\delta, 0)$, there exists $y \in (x, 0)$ such that $\dfrac{f(0)-f(x)}{0-x} = f'(y)$; since $y<0$, $f'(y)>0$, and since $x<0$, $f(0)-f(x)>0$.

3. Define $f : (0,1) \to \mathbb{R}$ by $f(x) = \cos\left(\frac{1}{x}\right)$.

a) (4 pts) Prove that f is not uniformly continuous.

b) (3 pts) Is $S = \{x \in (0,1) \mid f(x) \leq 0\}$ a closed set in $\mathbb{R}$?

d) (3 pts) Is $f((0,1))$ sequentially compact?

a) The sequence $\left(\frac{1}{n\pi}\right)$ is Cauchy (it converges to 0). But $\left(f\left(\frac{1}{n\pi}\right)\right) = (\cos(n\pi)) = ((-1)^n)$ does not converge (subsequences (1), (-1) converge to $1 \neq -1$). A unif. cont. function takes Cauchy sequences to Cauchy sequences, so f is not unif. Cauchy.

b) $\left(\frac{1}{(2n+1)\pi}\right)$ is a sequence in S since $\cos((2n+1)\pi) = -1 < 0$. But $\frac{1}{(2n+1)\pi} \to 0 \notin S$, so S is not closed.

c) Since $|\cos(y)| \leq 1$, $f((0,1)) \subseteq [-1, 1]$.

Since $\frac{1}{\pi}, \frac{1}{2\pi} \in (0,1)$ and $f(\frac{1}{\pi}) = -1$, $f(\frac{1}{2\pi}) = 1$, $-1, 1 \in f((0,1))$. By the IVT, $[-1, 1] \subseteq f((0,1))$, so $f((0,1)) = [-1, 1]$, which is closed and bounded and thus sequentially compact.

4. For each $n \in \mathbb{N}$ define $f_n : [-1,1] \to \mathbb{R}$ by $f_n(x) = \frac{nx^2}{1+nx^2}$.

a) (3 pts) Find a function $f : [-1,1] \to \mathbb{R}$ such that $f_n \to f$ pointwise.

b) (3 pts) Does $f_n \to f$ uniformly?

c) (4 pts) Change the domain of f_n and f to $[\frac{1}{2}, 1]$. Now does $f_n \to f$ uniformly?

a) Define $f : [-1,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

If $x = 0$, $f_n(0) = 0 \to 0 = f(0)$

If $x \neq 0$, $\frac{1}{n} + x^2 \to x^2 \neq 0$, and $x^2 \to x^2$, so

$f_n(x) = \dfrac{x^2}{\frac{1}{n} + x^2} \to \dfrac{x^2}{x^2} = 1$

b) No. f_n is a rational function and thus continuous. If $f_n \to f$ unif., f would be continuous. But $(\frac{1}{n})$ is a sequence in $[-1,1]$ converging to 0, while $f(\frac{1}{n}) = 1 \not\to 0 = f(0)$, so f is not continuous.

c) Let $\varepsilon > 0$. Choose $N = \frac{4}{\varepsilon}$.

If $x \in [\frac{1}{2}, 1]$, $n > N$

$$\left| f_n(x) - f(x) \right| = \left| \frac{nx^2}{1+nx^2} - 1 \right| = \left| \frac{1}{1+nx^2} \right|$$

$$\leq \frac{1}{nx^2} \leq \frac{4}{n} < \frac{4}{N} = \varepsilon$$

since $x \geq \frac{1}{2}$

So $f_n \to f$ uniformly.

5. (10 pts) Define $f : [0,2] \to \mathbb{R}$ by $f(x) = 1$ if $x \neq 1$ and $f(1) = 2$. Prove that f is integrable and find $\int_0^2 f$. Use only the definition of the integral and the following theorem:

$f : [a,b] \to \mathbb{R}$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon.$$

Let $\varepsilon > 0$. Choose $\delta < \frac{\varepsilon}{2}$.

Define a partition of $[0,2]$

$P = \{0, 1-\delta, 1+\delta, 2\}$.

Then $U(f,P) = 1(1-\delta-0) + 2((1+\delta)-(1-\delta)) + 1(2-(1+\delta))$

$= 1 - \delta + 4\delta + 1 - \delta$

$= 2 + 2\delta$

$L(f,P) = 1(1-\delta-0) + 1((1+\delta)-(1-\delta)) + 1(2-(1+\delta))$

$= 2$

Then $U(f,P) - L(f,P) = 2\delta < \varepsilon$. This proves f is integrable.

Thus, for all $\delta > 0$, $2 = L(f,P) \leq \int_0^2 f \leq U(f,P) = 2 + 2\delta$

so $\int_0^2 f = 2$.

6. Let $S = [0,1] \cup [2,3]$ and let $T = [0,1)$.

a) (2 pts) Prove that S is not connected.

b) (3 pts) Let $f : S \to \mathbb{R}$ be continuous such that $f(1) = f(2)$. Prove that $f(S)$ is connected.

c) (2 pts) Prove that T is not sequentially compact.

d) (3 pts) Let $g : T \to \mathbb{R}$ be continuous such that

$$\lim_{x \to 1} g(x) = g(0).$$

Prove that $g(T)$ is sequentially compact.

6) a) Let $A = (-\infty, \frac{3}{2})$, $B = (\frac{3}{2}, \infty)$.

Then $S \subset \mathbb{R} \setminus \{\frac{3}{2}\} = A \cup B$

$A \cap B = \emptyset$, so $S \cap A \cap B = \emptyset$

$S \cap A = [0,1] \neq \emptyset$

$S \cap B = [2,3] \neq \emptyset$.

b) Since f is continuous and $[0,1]$ is an interval, and thus connected, $f([0,1])$ is connected. Similarly, $f([2,3])$ is connected. Since $f(1) = f(2) \in f([0,1]) \cap f([2,3])$, $f(S) = f([0,1]) \cup f([2,3])$ is connected

c) $(1-\frac{1}{n})$ is a sequence in T, but $1-\frac{1}{n} \to 1$, so any subsequence converges to $1 \notin T$. So $(1-\frac{1}{n})$ does not have a subsequence converging to an element of T.

(i.e., T is not closed so not seq. compact).

d) g extends to a continuous function $\tilde{g} : [0,1] \to \mathbb{R}$, $\tilde{g}(x) = g(x)$ for $x \in T$,

$\tilde{g}(1) = g(0)$.

thus $\tilde{g}([0,1])$ is sequentially compact,

but since $\tilde{g}(1) = g(0)$, $\tilde{g}([0,1]) = g(T)$.

7. Consider the power series

$$\sum_{n=0}^{\infty}(n+1)\cos\left(\frac{n\pi}{2}\right)x^n$$

a) (4 pts) Prove that the power series coverges pointwise to a continuous function f on the interval $(-1,1)$.

b) (2 pts) Find $f''(0)$.

c) (4 pts) Find $\int_0^{1/2} f$.

a) $\sum_{n=0}^{\infty} (n+1)\cos\left(\frac{n\pi}{2}\right)x^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right) x^{n+1}$.

$\limsup \left|\cos\left(\frac{n\pi}{2}\right)\right|^{1/n} = \limsup \{1,0,1,0,1,0\ldots\} = 1$

so radius of convergence is $R=1$. A power series converges pointwise to a continuous function on $(-R,R)$

b) Since the radius of convergence is >0,

$f''(x) = \sum_{n=2}^{\infty} n(n-1)(n+1)\cos\left(\frac{n\pi}{2}\right)x^{n-2}$

and $f''(0) = 3\cdot 2\cos(\pi) = -6$.

c) Since $R > 1/2$,

$$\int_0^{1/2} f = \sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right)\left(\frac{1}{2}\right)^{n+1}$$

$= 0$ if n odd
set $n = 2k$

$$= \sum_{k=0}^{\infty} \cos(k\pi)\left(\frac{1}{2}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}\right)^{2k+1}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^k \qquad \left|-\frac{1}{4}\right| < 1, \text{ geometric series}$$

$$= \frac{1}{2}\left(\frac{1}{1+\frac{1}{4}}\right) = \frac{1}{2}\left(\frac{4}{5}\right) = \frac{2}{5}.$$